NASA TECHNICAL TRANSLATION NASA TT F-11,590

SYNTHESIS OF THE ANALYTICAL RESULTS ON OPTIMAL TRANSFER BETWEEN KEPLERIAN ORBITS

NASA TT F-11,590

Christian Marchal, Jean-Pierre Marec and C. B. Winn

Translation of "Synthese des resultats analytiques sur les transferts optimaux entre orbites Kepleriennes" Office National d'Etudes et de Recherches Aerospatiales, T.P. 515, 1967, 35 p., Paper presented at the 18th Congress of the International Astronautical Federation, Belgrade, 25-29 Sept. 1967

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 APRIL 1968

SYNTHESIS OF THE ANALYTICAL RESULTS ON OPTIMAL TRANSFER BETWEEN KEPLERIAN ORBITS

Christian Marchal, Jean-Pierre Marec, ONERA,
Chatillon-sous-Bagneux (Hauts de Seine), France and
C. B. Winn, Colorado State University

/1*

ABSTRACT; The optimization of the final mass depends upon the propulsion systems which are classified into two main categories: "high thrust systems" (S_1) or, better, systems with an imposed exhaust velocity (chemical propulsion, nuclear rockets, etc...) and "low thrust systems" (S_2) or, better, systems with an imposed exhaust power (nucleo-electric propulsion).

The analytical resolutions utilize the classical optimization theories [1] or more recent ones [2-5] with the method of Hohmann [32] or of Lawden [16] or of Contensou and Breakwell [5, 63]. They are classified into the "time free" case [32-77], the "time-fixed" case (close orbits) [78-129] and the studies of the "singular arcs" [189-196]; the time-fixed case between distant orbits [130-188] is almost always studied numerically.

1 - INTRODUCTION

The problem of optimal transfers and rendezvous is fundamental in space dynamics. Consequently, many studies have been devoted to these problems [1-226]. In very general terms, we can consider "transfer" (with rendezvous) a moving body M of variable mass subjected to the given gravitational field $\vec{g}$ and a thrust acceleration $\vec{\gamma}$. Every change in position or vector $\vec{r}$ and velocity $\vec{V}$ of this moving body occurs between two fixed instants t_o and t_f (Figure 1). /3

Optimization consists generally of minimizing its corresponding expenditure in propellant.

Propulsion systems are normally divided into two categories:

* Numbers in the margin indicate pagination in the foreign text.

1. "High-thrust" systems (transitional chemical engines, nuclear rockets) for which the maximum thrust F_{max} is sufficiently large compared to the local attraction force for the powered orbits to be of very short duration (quasi-impulses). The ejection velocity W is moderate (3-8 km/sec);

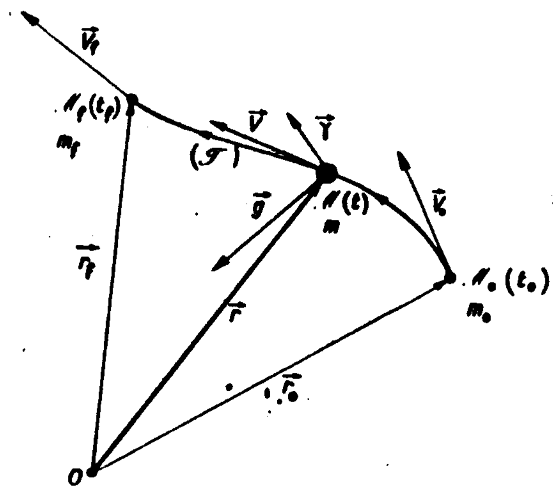


Figure 1. Transfer

2. "Low-thrust" systems (electrical motors for which $\gamma/g \sim 10^{-4}$ to 10^{-3} and which require long-lasting powered arcs. Their ejection velocity is large (10 km/sec and more).

From the viewpoint of a mathematical analysis of the problem, it is preferable to adopt a classification of the propulsion systems based on the capability of modulating the ejection velocity and of essentially distinguishing between the following two models:

1. (S_1) systems to which an ejection velocity is applied and whose thrust is limited (Figure 2a). ("High-thrust" system and electrical motors for which the ejection velocity is imposed).

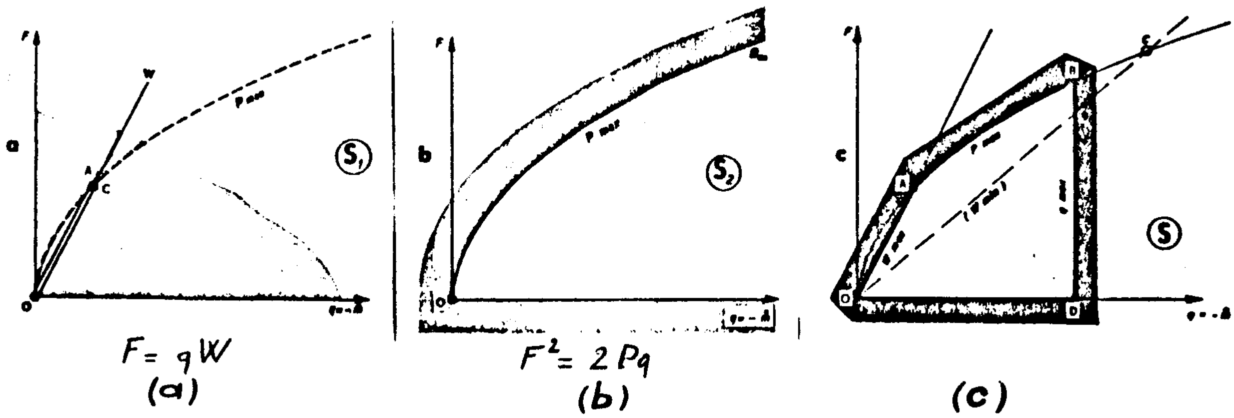


Figure 2. Propulsion Systems

We can then assume as an index of performance for minimizing the charac- /4
teristic velocity:

$$C_f = \int_{t_0}^{t_f} \gamma \, dt$$

2. (S_2) systems for which the ejection velocity is subject to modulation and where the sole limitation affects the power P of the jet (Figure 2b)

(idealized electrical motors). The performance index to be minimized may be assumed to be:

$$J_f = \int_{t_0}^{t_f} \gamma^2 dt$$

Studies on optimal transfer of more complex models, for example with limitations regarding the power of the jet, the ejection velocity and the flow (Figure 2c) [14, 91], or in the case of coupled motors of the (S_1) and (S_2) types [188] are rare.

The analytical solution of the optimal transfer problem based on the traditional optimization theories (Euler-Lagrange) [1], or more recent ones (Contensou, Pontryagin) [2-5], is very difficult and many studies use numerical optimization methods (gradient, dynamic programming, etc.) [6-8].

However, some preliminary analytical results were obtained [9-21] in the case of any given gravitational field, bringing to light the differences between the solutions corresponding to the various propulsion systems: for the (S_2) systems, the thrust is modulated during the transfer; for the (S_1) propulsion systems, there is a succession of maximum thrust arcs (except in a singular case) and of ballistic arcs.

All these solutions are based on the same notion of "vector efficiency" (Primer vector by Lawden [16]) indicating the optimum direction of the thrust whose geometrical interpretation can be given [91]: this vector has as its origin the moving body *M* and as its extremity a "pilot" moving body *P*, close to *M*, subject to the same thrust acceleration and to the same gravitational field as *M*, describing a "directrix curve" (*D*) which resembles the transfer trajectory (*T*) of *M* (Figure 3).

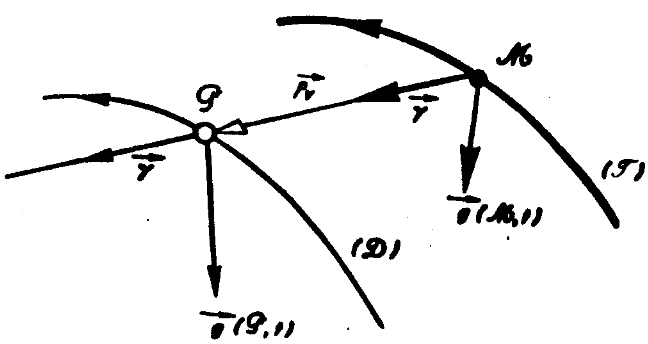


Figure 3. Pilot Moving Body and Directrix Curve.

A general analytical study was pursued more successfully in the zero [22-25] or constant [26] gravitational field. /5

2. <u>CENTRAL GRAVITATIONAL FIELD</u>

In the case of a central gravitational field, the general analytical study remains quite difficult [27-31]. Progress was recently made by using keplerian orbital elements (*O*), osculating with respect to the trajectory (*T*) of *M* at each instant, as components of the "state" instead of elements $\vec{r}$ and $\vec{v}$ or others used earlier. This change in variables was easily accomplished by using the notion of canonical transformation [29, 30].

The analytical results most frequently encountered were obtained in the three following specific cases:

1. Transfer of undefined duration [32-77].
2. Transfer of fixed duration between close orbits [78-129].
3. Singular arcs [189-196].

In contrast, the study of fixed-duration transfers between distant orbits [130-188] requires almost always the use of numerical methods.

These are the cases which will now be examined in succession.

2.1 - Transfers of Undetermined Duration /6

This chapter is developed at length in [74].

Many studies have been made by assuming the duration of the transfer to be undetermined. This is a realistic assumption in the case of short-duration transfers (maneuvers near a planet); on the other hand, the calculations are then very much simplified because the time is an inconvenient parameter in celestial mechanics.

The S_2-type propulsion systems (ejection power imposed), in those cases lead to solutions of indefinite duration and infinitely low cost. Therefore, only a study of the S_1-type propulsion systems is of interest (ejection velocity imposed).

The impulse was first studied by a simplified method which may be called the "Hohmann method": *a priori* we assume the number of impulses to be fixed and calculate them so that their sum is minimal [32-62].

The use of the "Contensou method" [3-5] has permitted the determination of the truly optimal transfer in many specific cases.

Very few analytical studies [48, 69] were devoted to the case of transfers between hyperbolic orbits. The initial and final conditions in this case are always applied at an infinite distance (with the exception of studies regarding the "singular arcs": cf. chapter 2.3) and the possibility of producing impulses is always present (losses due to limitations of thrust are then very small in the ordinary cases).

Not taking into account the possibility of braking or maneuvering in the atmosphere, optimal solutions can be of two types:

1. Transfer "by the parabolic (energy) level" (Figure 4) with six impulses

from the I_1 to I_6. There are two intermediate near-parabolas with their perigees grazing I_2 and I_5; impulses I_3 and I_4 which are used to make the nearly-parabolic change are infinitely distant and infinitely small.

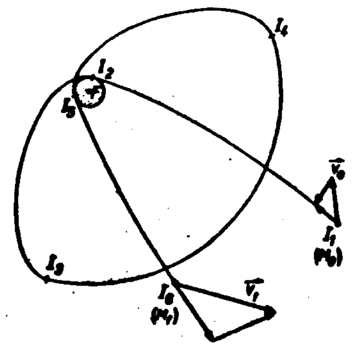


Figure 4. Transfer "By the Parabolic Level" (or Six-Impulse Transfer).

2. "Remote transfer" with two initial impulses (one at the initial point, the other at the final point); the moving body never passes at a finite distance from the center of attraction. /7

3. "Close transfers": the moving body passes at a finite distance from the planet of attraction. These transfers, covering nine different types, are all in the same plane and have a maximum of four impulses of which at the most one is at a finite distance from the planet. Seven of the nine types include a pass which comes close to the surface of the planet.

Note: If the initial and final conditions are not only assumed to be at an infinite distance but also such that the initial and final velocities $\vec{V}_0$ and $\vec{V}_f$ are vertical (the first one descending and the second one ascending) -- conditions which quite frequently are only planned -- the "remote transfer" is never optimal and the other types of transfer never have more than two impulses which are not infinitely small.

Optimal orbit ascents also have been the subject of very few analytical studies [76, 77]; so far they have always led to impulse solutions (almost always bi- or tri-impulse solutions).

This is the case of transfers between elliptical orbits which are not secant with respect to the planet of attraction which has been by far, the one subject to the most extensive studies. We will examine it now in detail:

Optimal transfers between elliptical orbits not secant with respect to the planet of attraction [32-75, except 48 and 69].

The possibility of breaking down the powered arcs into elements described in one single interval turn permits us to revert always to a study of the impulse case.

It has been demonstrated that the optimal solutions never include singular arcs [196] and it is likely that they are always either of the mono-, bi-, or tri-impulse type, or of the biparabolic type (Figure 5), with two finite impulses tangential to the perigees and two impulses infinitely small at large distances (to pass from one quasi-parabola to another). This latter type especially is always optimal if the angle formed by the initial and fianl orbit planes is greater than 60.1850°.

The following are the three most important cases studied analytically: /8

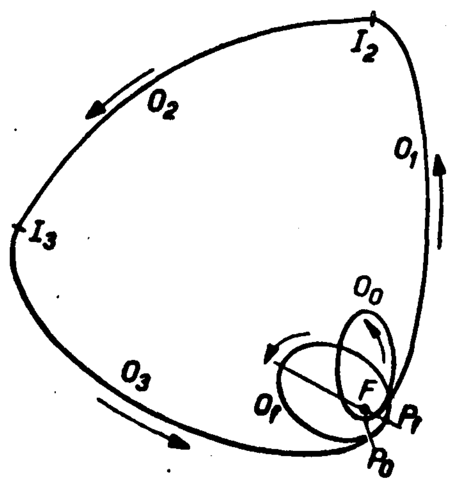


Figure 5. The Biparabolic Transfer.

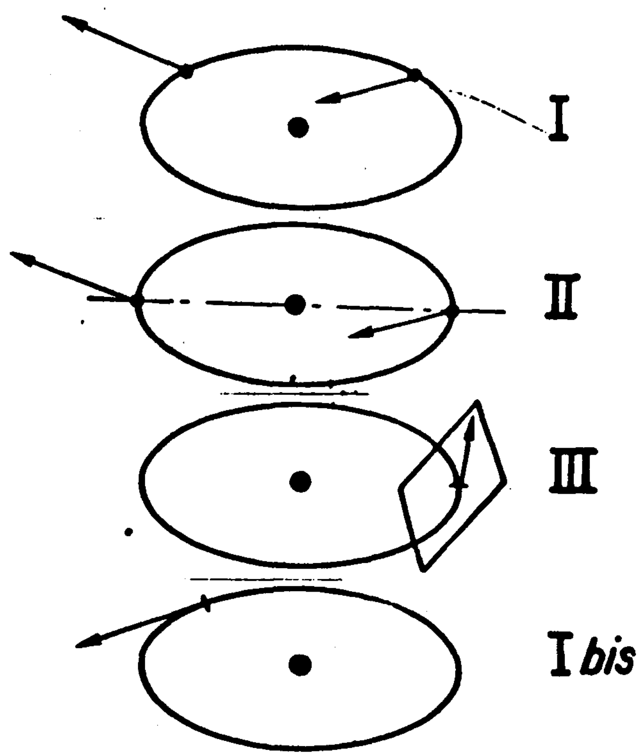


Figure 6. The Four Optimal Transfer Types Between Near-circular Close Orbits.

I. <u>Optimal transfers between nearly circular close orbits [70,73,91,93,94,95]</u>.

This case is evidently of very great practical importance. The first-order study leads to solutions of the following four types (Figure 6):

Type I: Bi-impulse type (two well-determined impulses)

Type II: Bi-impulse nodal type (two well-determined impulses applied to the nodes and symmetrical directions with respect to the orbit plane)

Type III: Singular tri-dimensional type. This is a degenerate type: A certain choice is available in the position of optimal impulse, but the optimal direction of the thrust is always well determined; it is always in one of the two planes containing the tangent and inclined 30° with respect to the local horizontal.

Type I bis: Singular plane. This type corresponds to Type I in the case of a plane. The thrust is tangential, but the optimal point of application of the thrust is no longer determined; this is a second case of degeneracy.

The first-order study in the case where the eccentricity of the orbits is low, but yet large with respect to the magnitude of the transfer [91, 94] and the second order study [73] eliminates the degeneracies and leads to bi- or tri- impulse solutions.

II. <u>Transfers between coplanar elliptical orbits</u>.

This case has been almost totally investigated.

A study of the area of maneuverability [3-5] leads to the notion of the /9
"useful angle" (Figure 7) outside of which any thrust or impulse is nonoptimal. This angle, always consisting of two angles opposed by the summit and which are located within the acute angle of the tangent and the local horizontal, always occupies less than 20% of these acute angles; today it is known with great accuracy [65,66,68,70,72,74,75] and does not depend on the eccentricity e of the osculating orbit and the true anomaly v of the point studied:

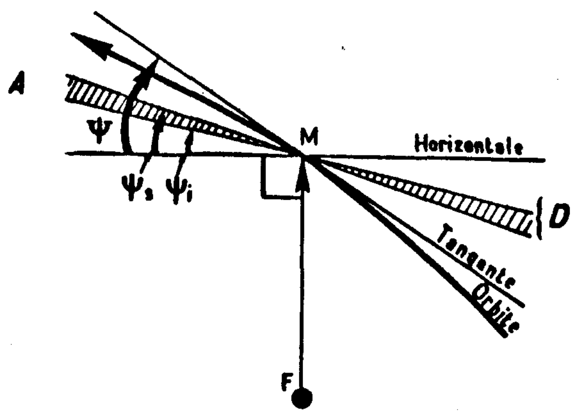


Figure 7. The Useful Angle.

$$\mathrm{tg}\,\psi_i = \frac{e \sin v}{3 + e\cos v} + \frac{18 e^3 \sin^3 v}{(3 + e \cos v)^5} + \frac{2}{81} e^5 \sin^5 v + \mathrm{ordre}\left(e^6 \sin^5 v\right)$$

(this expression is not always applicable to $e > 0.925$, a value starting at which the area of maneuverability undergoes a major qualitative change [66,68]).

$$\mathrm{tg}\,\psi_s = \frac{(2-x)\, e \sin v}{4 + x - x^2} - \frac{2(6-x^2)^2 e^3 \sin^3 v}{(3+x)(4+x-x^2)^4} - \frac{13\, e^5 \sin^5 v}{1536} + \mathrm{ordre}\left(e^6 \sin^5 v\right)$$

where $x = e \cos v$ (the second term contains $e^3 \sin^3 v$ and not $e^3 \sin v$ as shown in [74] by error).

The commutations which determine the succession of optimal impulses come about whenever the sides of the useful angle coincide with the thrust direction; they are known very accurately [66,70,72].

We will use the letter A to designate an accelerating impulse (drawn ahead of the useful angle, Figure 7) and the letter D to designate a decelerating impulse.

The optimal transfers include eight possible types: A - D - AA - AD - DD - - AAD - ADD and biparabolic (Figure 5).

The one-impulse modes (A or D) are rare because of the smallness of the useful angles and the three-impulse modes (AAD or ADD) apply only to very large eccentricities: it is necessary that the sum of the eccentricities of the initial and final orbits be greater than 1.712 (for a more accurate description, see the "Moyer domain" and conditions regarding the other elements in [74]).

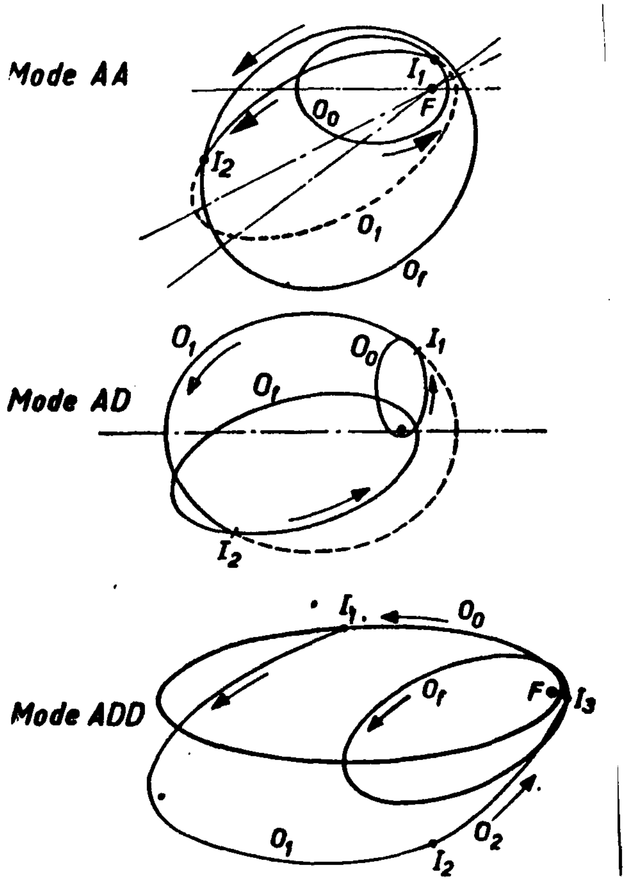


Figure 8. Coplanar Orbits -- Examples of Optimal Transfers.

On the other hand, the biparabolic mode is of little practical interest between coplanar orbits: the physically realizable solutions which come close generally require considerable distances in order to be more economical than the best of the one- or two-impulse solutions. In practice, therefore, optimal transfer is almost always a one- or two-impulse solution; one may even add the following: it is almost always of the AA mode (Figure 8) if the final orbit O_f encircles the initial orbit O_o of the mode AD (Figure 8) if the two orbits are secant, and of the DD mode, if O_f is contained within O_o. /10

Certain specific cases are very simple such as, for example, that where the two orbits O_o and O_f are equal (the optimal solutions, if they are not of the "biparabolic type" are of the "AD symmetrical" type, (Figure 9) or that where the two coplanar orbits have the same major-axis direction (the specific two-impulse or biparabolic case of the situation described below).

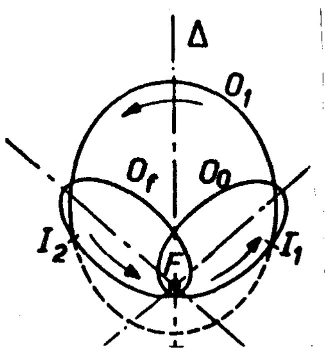


Figure 9. Equal Coplanar Orbits -- Symmetrical AD-Type Transfer.

III. Transfers between coaxial orbits (coplanar and noncoplanar).

Two elliptical orbits will be called coaxial if they have the same major-axis direction. We distinguish between direct-coaxial orbits (perigees on the same side) and inverse coaxial orbits.

These transfers, whose study was begun in [33,40, 43,56,57,60,65,71,75] were completely analyzed by C. B. Winn [71 bis]; they contained as a specific case transfers between circular orbits. /11

The optimal solutions use intermediate orbits which are coaxial with the initial orbit O_o and the final orbit O_f. They are of the following types:

1. Biparabolic type (Figure 5). This mode may be considered of the tri-impulse type if O_o and O_f are direct-coaxial orbits.

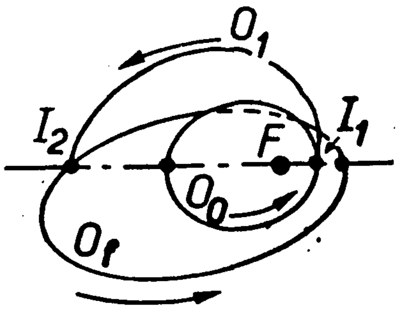


Figure 10. Direct or Aligned Coaxial Orbits -- The "Generalized Hohmann Transfer".

2. Two-impulse type solutions: between direct-coaxial orbits the transfer is of the "generalized Hohmann type" (Figure 10) with an intermediate ellipse whose apogee is at the highest apogee and whose perigee is the perigee of the other orbit.

Between inverse coaxial orbits the transfer is performed either "by perigees" or "by apogees" (Figure 11).

3. Three-impulse type (Figure 12).

The three impulses occur on the axis at I_1, I_2, and I_3. There are two intermediate ellipses whose axes are I_1I_2 and I_2I_3. If O_o and O_f are direct coaxial ellipses, I_1 and I_3 are their perigees and those of the intermediate ellipses. I_2 is the common apogee of the latter and is farther removed than the apogees of O_o and O_f.

Determination of the planes of intermediate ellipses and of position I_2 is similar to an optical problem. /12

If O_o and O_f are circular orbits, Figure 13 shows the optimal transfers mode as a function of the ratio r of the radii (assumed ≤ 1 if necessary, due to reversibility) and of the angle i of the orbital planes.

The remarkable study by C. B. Winn [71 bis] permits a determination in each case of the optimal transfer mode and the position of the intermediate ellipse(s). Following are examples of diagrams of cases where the initial orbit eccentricity is zero followed by 0.1.

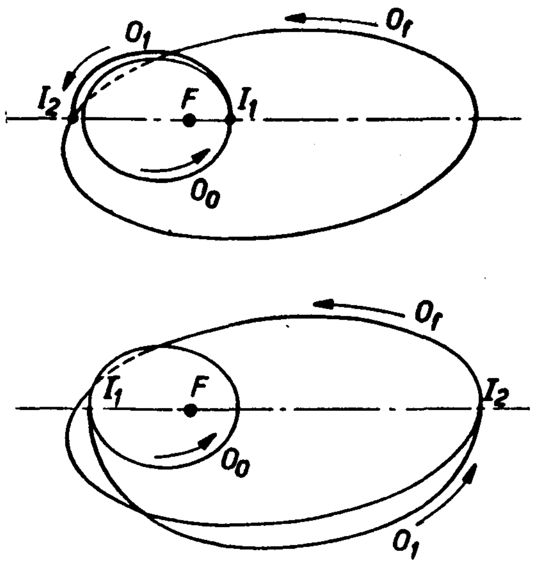


Figure 11. Inverse or Opposing Coaxial Orbits. The Transfers "By the Perigees" and "By the Apogees".

1. Disposition of the graphics (Figure 14). $P_o (= a_o(1-e_o))$ is the distance to the center of the perigee of the initial oribt; likewise, $A_f = a_f(1+e_f)$ and $P_f = a_f(1-e_f)$ (assuming by

convention $e_f < 0$ while O_o and O_f are inverse coaxial orbits).

2. Case where $e_o = 0$

i is the angle formed by planes O_o and O_f

2 : two-impulse transfer

3 : three-impulse transfer

BP : biparabolic transfer

3. Case where $e_o = 0.1$ /13

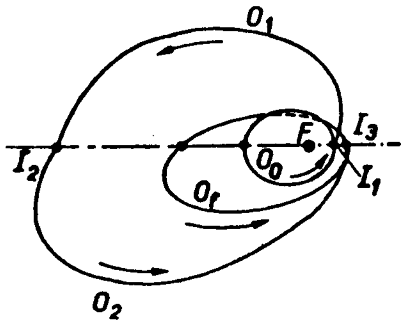


Figure 12. The Three-impulse Transfer.

Other cases were studied analytically such as the infinitesimal rotation of the orbit plane [87] as well as transfers between orbits, one of which has its eccentricity close to 1 [74, 75]; they always lead to solutions which never have more than three finite impulses. Furthermore, optimal hyperbola-ellipse transfers are easy to obtain [74].

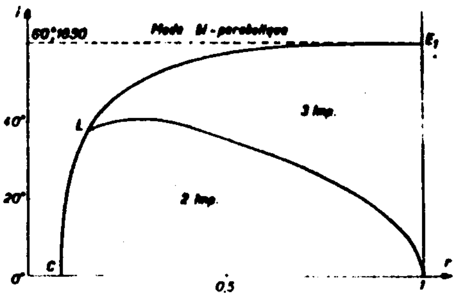


Figure 13. Optimal Transfer Type Between Circular Orbits; r = = Radius *ratio*.

From the practical viewpoint, the accomplishment of theoretical impulses by way of one or more maximal thrust arcs (if there are several of them, they are one rotation apart) leads to solutions [161] hardly more costly than the impulse solutions. Moreover, the performance of rendezvous at the same price as simple transfer has been accomplished by a very simple and elegant solution [78, 213, 226] by using an intermediate orbit of the proper period. Finally, the practical solutions similar to biparabolic transfers (Figure 5) are of great interest if the angle i is large.

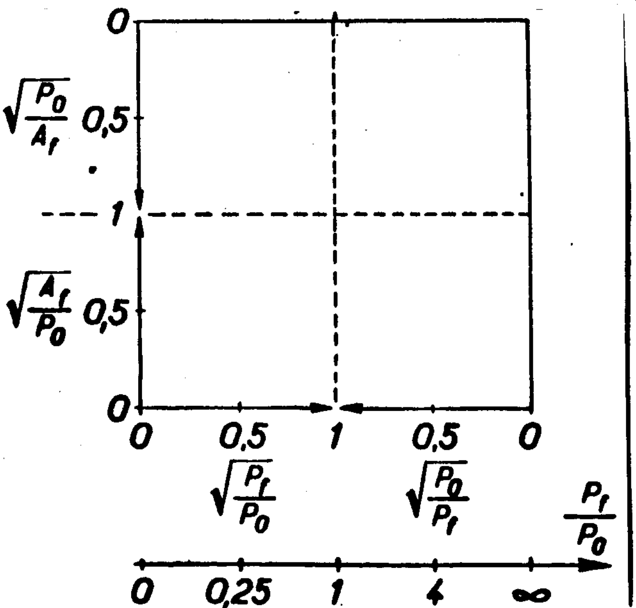


Figure 14. Disposition of the Graphics.

2.2 - Transfers of fixed duration

2.2.1 Close orbits.

The problem of transfers of fixed duration between close orbit orbits is well suited for analysis The hypothesis of small distances (of the order of ε) of the osculating orbit during the transfer, with respect to a nominal reference orbit, permits the linearization of the problem. The selection of the orbital elements as the coordinates of state leads to a particularly simple adjunct which is a linear function of time [81, 87] or even constant [91], if the sixth component is properly chosen.

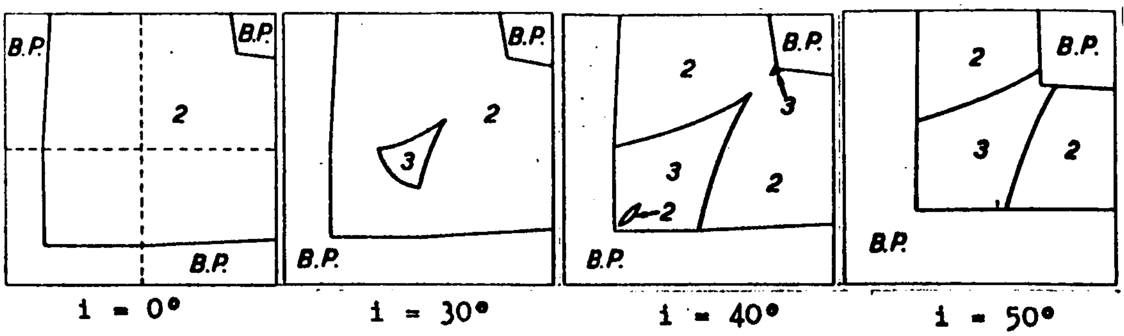


Figure 15. Case Where e_o = 0.

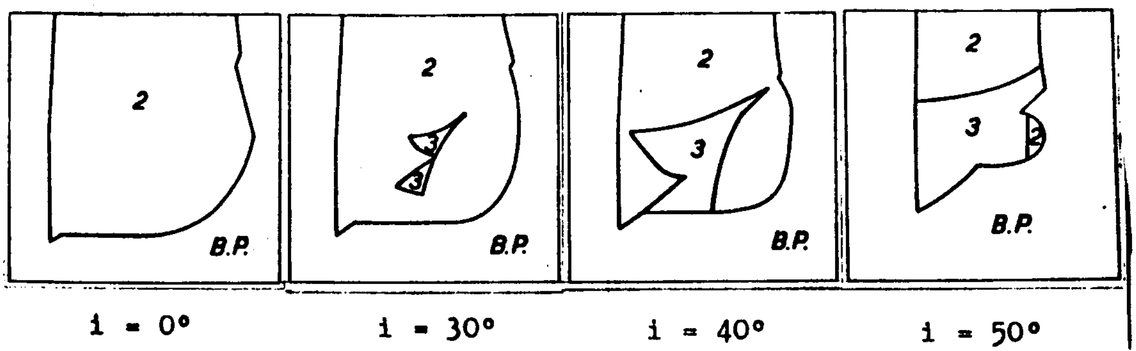


Figure 16. Case Where e_o = 0.1.

The use of the "efficiency curve" instead of the pilot moving body P in mobile axes (first Lawden locus [16]) is of great interest.

The solution is much simpler in the case of (S_2) propulsion systems, where /14 the first-order general solution can be obtained [80, 81] and where, furthermore, the transfer problem between circular close orbits [82] followed by that of rendezvous between nearly circular close orbits [83] has been studied in detail.

In the case of (S_1) propulsion systems, the law of all-or-nothing thrust introduces nonlinearities even in the first-order study. Few general results can be obtained.

Meanwhile, let us quote the following results [91]:

In the case of transfers between any given elliptical close orbits, there can be at the most three arcs of maximum thrust (or, at the limit, three impulses) per revolution.

In the case of transfers between nearly circular orbits of eccentricity $e \leq$ order of $\varepsilon << 1$, except in a singular case, there are never more than two arcs of maximal thrust (or two impulses) per revolution.

Note: the two preceding results assume the duration of the transfer to exceed one revolution.

The singular cases (of the linearlized problem) where the solution is no longer unique but degenerates into a large number of solutions (with a certain amount of liberty in the choice of position of the points of application and the magnitude of thrust) can only occur for transfers between nearly circular orbits ($e \leq$ order of $\varepsilon << 1$). In this case, there are two types of singular solutions: type I bis (singular plane) and type III (singular tridimensional) already discussed in section 2.1.

It is also possible to obtain results concerning the phenomena of induction (nonimposed variations of certain orbital elements, induced by the imposed variations of other orbital elements) which generalize, in the case of (S_1) propulsion systems, the results concerning the decouplings encountered for (S_2) propulsion systems [81].

There is mutual noninduction between the rotation of the orbital plane and the modifications in the orbital plane, but there is no decoupling between the two problems.

Similar results are obtained if the number of revolutions is an integer (or for a large number of revolutions). The analytical study may be extended to certain particular transfer classes which are of evident practical interest [91].

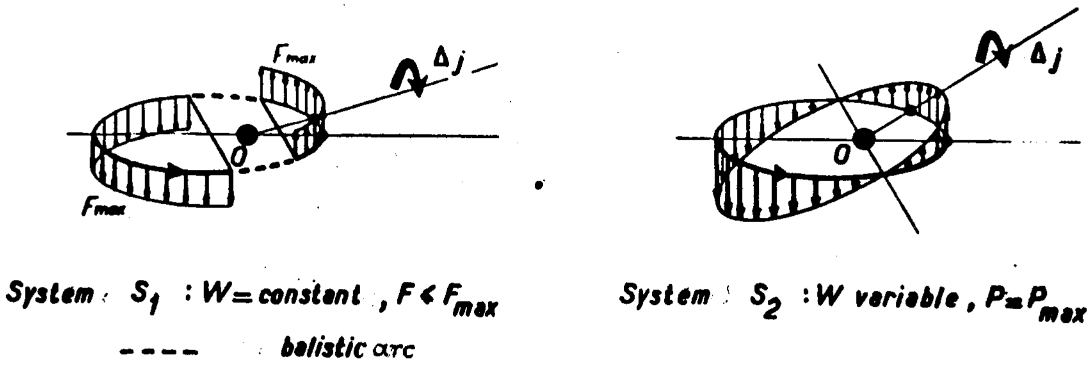
The complexity of the study of these particular cases increases as the /15
number of orbital elements increases whose variation is imposed, although, in contrast, simplified hypotheses must be formulated on an increasing scale.

1. Infinitesimal optimal "dilatation" of the semi-major axis may be dealt with in the elliptical case for an integral number (or a large number) of revolutions. The maximal thrust is tangential and applied to each revolution over an arc which circles the perigee.

2. Infinitesimal optimal rotation of the plane is also discussed in the same hypotheses. The maximal thrust is normal to the orbital plane and applied to one or two arcs during each revolution [84, 87] (Figure 17a).

3. Optimal transfers between coplanar circular close orbits were studied for any given transfer angle [85-91].

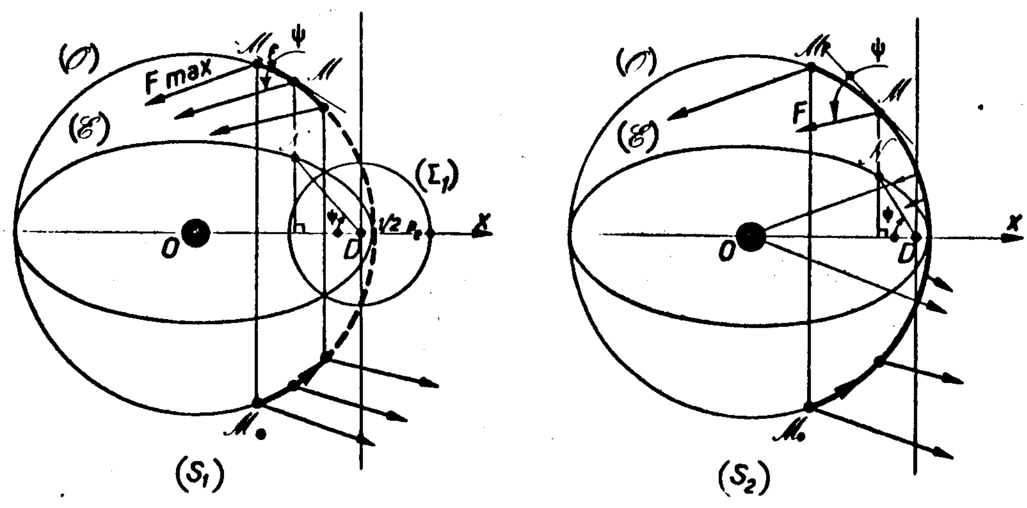
A distinction must be made between the regular and the singular solutions (of the linearized problem). The latter can result only for transfer angles above 180°.



a b

Figure 17. Infinitesimal Rotation of the Orbital Plane.

The regular solutions include an alternation of maximal thrust and ballistic arcs (Figure 18a). Optimal thrust is antisymmetric with respect to the axis of symmetry of the transfer arc. The law of thrust orientation and consumption depends not only on the transfer angle but also on the maximal thrust available.



a b

Figure 18. Transfer Between Circular Coplanar Close Orbits.

The singular solutions are of the type I bis (singular plane) and /16 correspond to a tangential application of the thrust. This is a degeneration of the linearized solution. In fact, it is sufficient to divide the thrust along the transfer arc in such a way that the "dilatation" desired is obtained and that the "center of mass" of the partition be at the center of the orbit. These singular solutions are particularly important because the specific "dilatation" is then maximal (in the linearized study) (Figure 19).

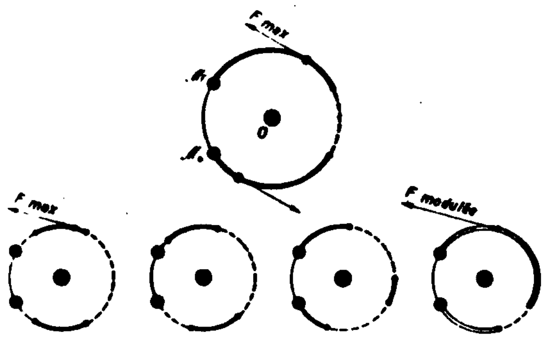


Figure 19. Hohmann-type Solution and Degenerated Solutions.

This degeneration disappears if we make a study at higher orders than those of singular solutions [86, 89].

4. This higher-orders study could be generalized to cover the cases of optimal transfer planes of the Hohmann type between nearly circular, coaxial-direct, and nonsecant close orbits [90, 91].

These transfers are accomplished /17
economically by using the two-impulse Hohmann solution if the thrust of the motor is not limited (Figure 20). Otherwise, they consist of a succession of arcs of maximal thrust of slightly decreasing duration starting with the perigee and moving toward the apogee (Figure 21).

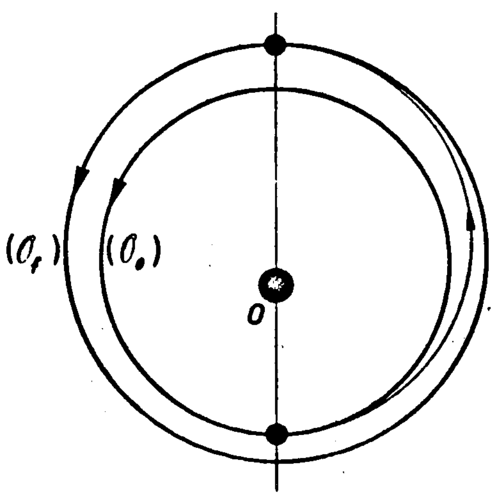


Figure 20. Hohmann Transfer.

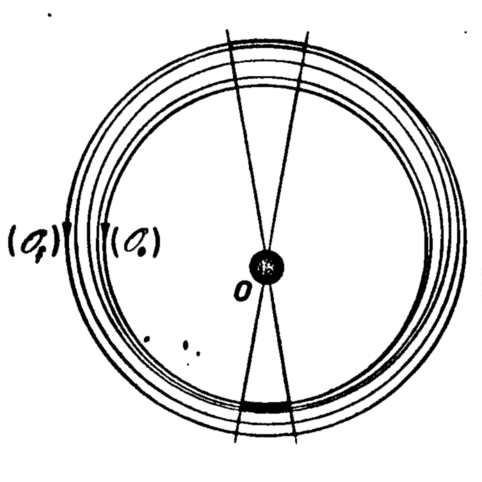


Figure 21. Limited Thrust Case -- Third Order Study.

Optimal thrust is applied in practice according to a bisector of the angle formed by the local horizontal and the tangent to the trajectory.

The difference between the characteristic velocities (integral of the thrust acceleration) of such a solution and that of the Hohmann solution relative to the same transfer is of the third order with respect to the magnitude of the transfer, even for arcs of relatively long-lasting thrust. This explains the degeneration noted in the linearized study of such solutions.

5. The solution relative to the optimal long-lasting rendezvous between nearly circular close, coplanar or noncoplanar orbit [96], is simply derived from that pertaining to the transfers (Cf. section 2.1).

According to the rendezvous to be performed, the optimal solution is of types I, I bis, II, III (as in the transfer case) or, for distant rendezvous, of two new types: type IV, with four impulses (which can be reduced to three) and type IV bis, in a singular plane.

A convenient choice of the variable described and of the sixth component of state permits discussion of the rendezvous case where the initial angular shift is large, while the majority of the studies dealing with these problems discuss only rendezvous where this shift is in the order of ε [96-114] and, furthermore, often contains major simplified hypotheses [104-108, 111-113]. /18

The fixed-time transfer problem between close orbits is evidently tied to the problem of orbit corrections from their deterministic [115-120] or stochastic [121-129] viewpoint.

2.2.2 - Distant Orbits.

The fixed-time transfer problem between distant orbits is very difficult to solve.

For "high thrust" propulsion systems, the number of impulses assumed is generally fixed *a priori* and the impulses are determined in such a way that their sum is minimal [130-136].

For "low thrust" propulsion systems, the studies are essentially of a numerical order. Frequently, the command law is partly imposed (or indeed totally imposed, which eliminates any optimization!).

We thus distinguish among the following:

1. For (S_2) propulsion systems, the studies where the (modulable) thrust acceleration is optimal and where the thrust direction is tangential [137] or optimal [138-152].

The remarkable purely analytical study of long-lasting cases performed by Edelbaum [140] should be singled out.

2. For (S_1) propulsion systems, the studies where the thrust is optimal (succession of maximal thrust and ballistic arcs) and where the thrust direction makes a constant angle with the local horizontal [153], or is horizontal [154] or is optimal [155-162].

One particular case of this type of studies concerns transfers where maximal thrust is applied continuously.

Very often, in order to simplify the calculations, it is assumed that it is the thrust acceleration and not the thrust force which is constant. /19

The thrust direction is then assumed fixed [163], radial [164-168], tangential [169-174], normal [175], making a constant angle with the local horizontal [176-178], or optimal [179-187].

2.3. Singular arcs [189-196]

Singular arcs whose prototype is Lawden's spiral (Figure 22) [189], are found in studies of S_1 propulsion systems (ejection velocity imposed, cost parameter = characteristic final velocity C_f).

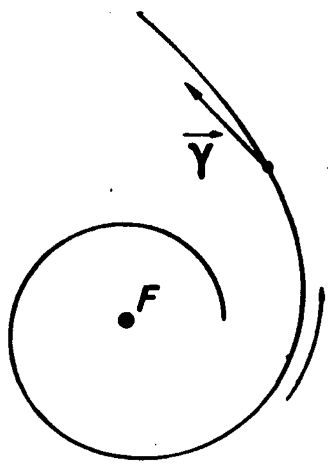


Figure 22. Lawden's Spiral.

To study them, let us use the Pontryagin theory [2] with the following parameters:

1. Parameter of state: $\vec{r}$, $\vec{V}\left(=\frac{d\vec{r}}{dt}\right)$, $C(t)=\int_{t_0}^{t}\gamma\,dt$

2. Command parameter: $\vec{\gamma}$ = acceleration due to the motors (hence $\frac{d\vec{V}}{dt}=\vec{\gamma}-\mu\frac{\vec{r}}{r^3}$); γ is subject to the conditions $0\leqslant\gamma\leqslant\gamma_{max}$, where γ_{max} is a given function of C.

3. Pontryagin parameters associated with $\vec{r}$, $\vec{V}$ and C: $\vec{p}_r$, $\vec{p}_V$, $\vec{p}_c$; $\vec{p}_V$, an efficiency vector, is Lawden's "primer vector".

From there, we obtain the Hamiltonian H: $H=\vec{p}_r\cdot\vec{V}+\vec{p}_V\cdot\left(\vec{\gamma}-\frac{\mu\vec{r}}{r^3}\right)+p_c\gamma$

Let us make $|\vec{p}_v|=p_v$; the maximization of H leads to:

I. $\vec{\gamma}$ is parallel to $\vec{p}_v$ and points in the same direction.

II. $H=\vec{p}_r\cdot\vec{V}-\mu\frac{\vec{r}\cdot\vec{p}_V}{r^3}+\max(0;p_v+p_c)\cdot\gamma_{max}$,

where: $\gamma=0$ si $p_v+p_c<0$, γ is of any magnitude if $p_v+p_c=0$, $\gamma=\gamma_{max}$ if

$p_v+p_c>0$.

In general, therefore, $\gamma = 0$ or γ_{max}. The singular arcs are obtained by selecting γ precisely in such a way that $P_v + P_c$ is zero all along the arc. They therefore meet Pontryagin's optimality conditions.

We must therefore set down that $P_v + P_c$ and its successive derivatives are /20
zero: we calculate the expression of these derivatives up to the point where one of them (in fact, the fourth one) contains γ explicity which permits determining the necessary acceleration value.

Clearly, a similar calculation can be made for any force field. The derivative of P_c is obtained easily: $\frac{dp_c}{dt} \equiv -\frac{\partial H}{\partial C} = 0$ donc $p_c = p_{c_f}(=-1)$

For P_v we obtain: 1. $p_v^2 = 1$; 2. $\frac{d}{dt}(p_v^2) \equiv -2\vec{p}_v\vec{p}_r = 0$

3. $\frac{d^2}{dt^2}(p_v^2) \equiv 2p_r^2 - \frac{2\mu p_v^2}{r^3} + \frac{6\mu}{r^5}(\vec{r}\cdot\vec{p}_v)^2 = 0$;etc.

We make $s = \frac{\vec{r}\cdot\vec{p}_v}{r} = \sin\varphi$ (φ is therefore the positive upward angle of the direction of $\vec{P}_v$ and of $\vec{\gamma}$ with the local horizontal plane.) From the equation

$\frac{d^2}{dt^1}(P_v^2)$ we obtain $P_r^2 = \mu\frac{1-3s^2}{r^3}$, therefore $s^2 \leq \frac{1}{3}$, therefore $|\varphi| \leqslant 35.264°$.

The integration of the plane singular arcs is easy [196]. They verify, in polar coordiantes r and θ, (Figure 23), the equation: $r = \frac{r_1}{1-3s^2}\left(s^3 + \frac{Hr^2}{3\mu}\right)^2$

and the quadrature: $\frac{d\theta}{d\varphi} = \frac{\mu(3s-4s^3) - Hr^2}{\mu s^3 - Hr^2}$

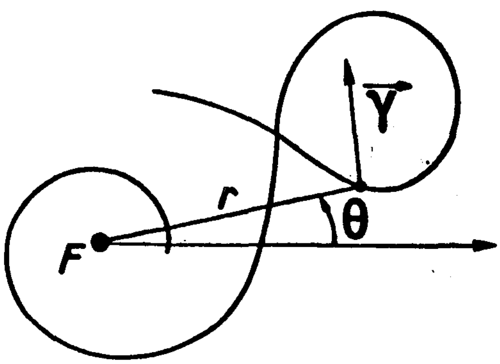


Figure 23. Planar Singular Arc.

(r_1 is of constant length).

For H = 0, the arc obtained is Lawden's spiral (Figure 22).

For $r_1 = +\infty$ we obtain the reversible arc (Figure 24), with a check point at A for $s^2 = \frac{1}{3}$ and a point B where γ becomes 0 (it is then necessary to change the sign of s and that of H in order to preserve the function $\gamma \geq 0$).

The only tri-dimensional integrated case is that of the "circular arc" /21

(Figure 25) whose semi-angle is at the center: $\text{Arc}\sin\frac{\sqrt{3}}{3}$ (= 35.264°).

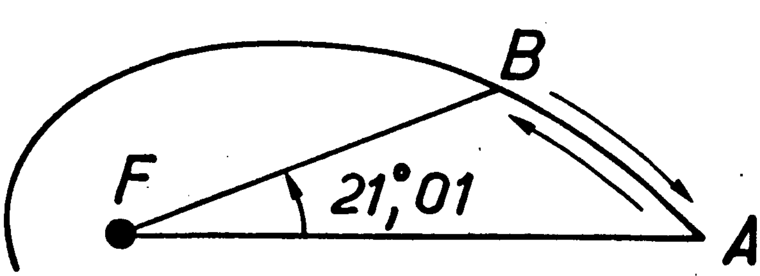


Figure 24. The Reversible Arc.

$$r = \sqrt{-\frac{3\,s^3}{H}} \qquad \theta = \theta_0 - \frac{\varphi}{4} - \frac{3\cot\varphi}{4}$$

In the other cases, the integration must be facilitated by using first integrals [27]:

1. H = constant;
2. $\vec{A} = \vec{r}\wedge\vec{p_r} + \vec{V}\wedge\vec{p_V}$ = = constant vector;
3. $K = C + 2\vec{r}.\vec{p_r} - \vec{V}.\vec{p_V} - 3Ht$ = = constant (the latter applies only to singular arcs and impulse transfers).

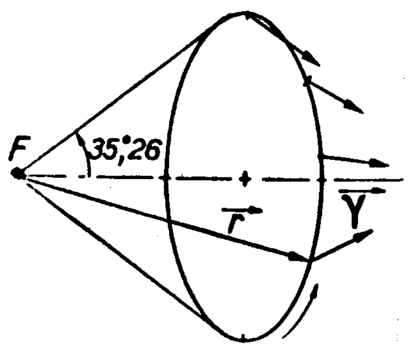


Figure 25. The Circular Arc.

$$r \equiv r_0;\ s = \frac{1}{3}$$
$$V = \sqrt{\frac{2\mu}{3r}}$$
$$\Upsilon = \frac{\mu}{r^2}$$

The singular arcs satisfy Pontryagin's optimality conditions, but these conditions are insufficient according to many studies dealing with the necessary and sufficient optimality condition [191, 194-196].

We thus obtain the following:

1. Every singular arc containing a portion for which s is positive is nonoptimal; particularly Lawden's spiral and the "circular arc" are not optiaml.

2. The same applies if the final time t_f is undetermined and if one of the dynamic conditions found along the arc studied corresponds to an elliptical orbit not secant with respect to the attraction planet: a major economy can be accomplished by making one or more turns around the center of attraction.

3. Let us consider a piece of singular arc on which s is always negative (for example, arc BAB, Figure 24) and let us make the following assumption for this sector:

$$r_m = \min r\ ;\ S_M = \max s\ (<0);\ V_M = \max V;\ J = \max\left(V_M; \sqrt{\frac{\mu}{r_m}}\right)$$

Every portion whose time is less than $\left(-\frac{S_M r_m}{8J}\right)$ of the piece of arc studied is optimal if one of the following cases applies:

A. If t_f is fixed.

B. If H = 0 and if t_f is subject to $t_f \leqslant t_0 + 2\sqrt{\frac{r_m^3}{\mu}}$.

C. If H = 0, t_f is undetermined, and if none of the dynamic conditions found along the trajectory of the portion under study corresponds to an elliptical orbit not secant to the planet of attraction (and provided that these ellipses are not accessible at a price lower than that of the portion studied).

The real limit time is probably much higher than $\left(-\frac{S_M r_m}{8J}\right)$.

The question of optimal junction of singular arcs to ordinary arcs (ballistic or maximal thrust) is also quite difficult except if $\gamma \to 0$ or γ_{max} at the extremity of the singular arc (or if $s \to 0$). Otherwise, there is an infinity of ballistic arcs and alternating maximal-thrust arcs immediately next to the extremity of the singular arc. Such a series is evidently difficult to analyze (but there are some which are optimal). In the impulse case, the singualr arc simply terminates with an impulse followed by a ballastic arc. /22

This study of adjoining orbits shows particularly well that singular arcs are found in the general cases and not in the particular cases (although they themselves are connected only with initial and final conditions which satisfy many given realtionships).

Contensou proposed an extension of the notion of the singular arc (to the case of transfer of undetermined duration of elliptical orbits): the optimal application point of the alternate thrust between two (or more) positions along the orbit and the transfer consists of an infinity of infinitesimal impulses located alternately at the various optimal positions (which requires therefore an infinity of orbits). Actually, such transfers were found which satisfy Pontryagin's conditions but, so far, none of them is truly optimal.

3. CONCLUSION

The analytical study of optimal transfers and rendezvous between keplerian orbits has made great progress in recent years, including the solution of particular cases (many cases are well-known at present), as well as from the general theoretical viewpoint [for example: optimal solutions make use of modulated-thrust arcs for type S_2 propulsion systems (ejection thrust imposed) and generally of ballistic arcs and maximal thrust arcs (as a result of the impulse limit) for type S_1 propulsion systems (ejection velocity imposed); very rarely, there are also singular arcs and cases of reticence (chattering arcs)].

Many general theoretical results should be obtained soon, such as, e.g., the maximal number of arcs of various types within an optimal solution (for S_1 systems: probably never more than one singular arc and, in the impulse case, six pulses. This maximum is clearly reduced to zero singular arcs and three impulses not infinitely small for transfers between elliptical orbits with undetermined duration).

Furthermore, it will be very interesting to study the possibilities of using secondary forces that exist in space, such as the equatorial oblateness of the planets, atmospheric braking and maneuvering and particularly the forces of attraction due to secondary bodies on trips in the solar system [197-211].

BIBLIOGRAPHY

Optimization Methods /23

1. Bliss, G.A.: "Lectures on the calculus of variations", The University of Chicago Press, Chicago, Ch. VII, 1946.

2. Pontryagin, L.S., Boltyanskii, V.G.; Gamkrelidze, R.V. and E.F. Mishchenko: "The Mathematical Theory of Optimal Processes", Interscience Publishers, John Wiley and Sons, Inc., New York, 1962.

3. Contensou, P.: "Note on the General Dynamics of the Powered Moving Body", Communication to the Technical Maritime and Aeronautical Assn., Vol. 45, No. 836, June 1946.

4. Contensou, P.: "Application of Methods and Mechanics of the Powered Moving Body to the Theory of Flight in a Plane", Communication to the Technical Maritime and Aeronautical Assn., Vol. 49, No. 958, 1950 Session.

5. Contensou, P.: "Theoretical Study of Optimal Trajectories in a Gravitational Field. Application to the Case of a Single Center of Attraction", Astronautica Acta, VIII, Parts 2-3, 1962.

6. Denham, Bryson: "A steepest ascent method for solving optimum programming problems", Journal of Applied Mechanics, June 1962.

7. Bellman, R.E. and Dreyfus, S.E.: "An application of Dynamic Programming to the determination of optimal satellite trajectories", J. British Interplanetary Soc., 17, 78-83, 1959-1960.

8. Bagaeva, N.Y. and Moiseyev, N.N.: "New methods for solution of problems of optimal transfer theory", 14th International Astronautical Congress, Paris, September 1963.

Any Given Gravitational Field

9. Lawden, D.F.: "Minimal trajectories", Journal of the British Interplanetary Society, 9, 179-186, July 1950.

10. Lawden, D.F.: "Minimal rocket trajectories", Jet Propulsion 23, 360-367 & 382, 1953.

11. Lawden, D.F.: "Fundamentals of Space Navigation", Journal of the British Interplanetary Society, 13, 87-101, 1954.

12. Breakwell, J.V.: "The optimization of trajectories", S.I.A.M. Journal, 7, 215-247, 1959.

13. Lawden, D.F.: "Interplanetary rocket trajectories", Advances in Space Science and Technology, Edited by F.I. Ordway, Academic Press, Vol. 1, New York, 1960.

14. Leitmann, G.: "Variational problems with bounded control variables, Optimization Techniques, Academic Press, Ch. 5, 1962.

15. Isaev, V.K. and V.V. Sonin: "On a non-linear problem of optimum control Problem formulation. Optimum control structure. First integrals", Revue russe: Automatismes et Télémécanique, 23, 1118-1129, 1962.

16. Lawden, D.F.: "Optimal trajectories for space navigation", Butterworths Mathematical Texts, Butterworths, London, 1963.

17. Ivanov: "Movement of a Body of Variable Mass with Limited Power and a Given Time of Powered Flight", Prikladnaia Matematika i Mexanika (Applied Mathematics and Mechanics), Vol. 27, No. 5, pp. 854-863, Sept.-Oct. 1963.

18. Lawden, G.H.: "Trajectory optimization for a rocket with a generalized thrust characteristic", Astronautica Acta, 10, 279-295, 1964. /24

19. Isaev, V.K.: "On certain features of the Mayer variational problem in rocket dynamics", Revue russe Automatismes et Telemecanique, 26, 1161-1168, 1965.

20. Isaev, V.K.: and Y. M. Kopnin: "A review of some qualitative results obtained in flight dynamics by an optimal processes theory", 17th International Astronautical Congress Madrid, October 9-15, 1966.

21. Robbins, H.M.: "An analytical study of the impulsive approximation", AIAA J., 4, 1417-1423, 1966.

Zero Field

22. Langmuer, D.B.: "Low thrust flight : constant exhause velocity in field-free space", Space Technology, ch. 9, John Wiley and Sons, 1959.

23. TSU, T.C.: "Requirements of interstellar flight", Astronautica Acta, VI, 247-255, 1960.

24. Spencer, D.F. and L.D. Jaffe: "Feasibility of interstellar travel", Astronautica Acta, IX, 49-58, 1963.

25. Anderson, G.M., Falb, P.L. and A.C. Robinson: "One dimensional minimum-time rendezvous for a thrust-limited rocket", AIAA J, 5, 1017-1019, 1967.

Constant Field

26. Isaev, V.K.: "The Maximum Principle by L. S. Pontryagin and Optimal Programming of the Thrust of a Rocket", Revue russe: Automatismes et Telemecanique, Vol. 22, No. 8, August 1961.

Central Field

General

27. Pines, S.: "Constants of the motion for optimum thrust trajectories in a central force field", AIAA J. 2, 2010-2014, 1964.

28. Lur'e, A.I.: "Thrust programming in a central gravitational field", IFAC Symposium on Automatic Control in the Peaceful Uses of Space, Stavanger, Norway, June 1965.

29. Fraeijs de Veubeke, B.: "Canonical transformations and the thrust-coast-thrust optimal transfer problem", Astronautica Acta, 11, 272-282, 1965.

30. Fraeijs de Veubeke, B.: "Optimal steering and cutoff-relight programs for orbital transfers", Astronautica Acta, 12, 323-328, 1966.

31. Edelbaum, T.N.: "How many impulses?", AIAA Paper No. 66-7, AIAA 3rd Aerospace Sciences Meeting, New York, January 24-26, 1966.

Undetermined Time*

32. Hohmann, W.: "The Attainability of Celestial Bodies", Oldenbourg, Munich, 1925. "The attainability of heavenly bodies", NASA Technical translation F.44, 1960.

33. Rider, L.: "Characteristic velocity changing the inclination of a circular orbit to the equator", ARS J.29, 48-49, 1959.

34. Smith, G.C.: "The calcualtion of minimal orbits", Astronautica Acta, V, 253-365, 1959.

35. Edelbaum, T.N.: "Some extensions of the Hohmann transfer maneuver", ARS J., /25
29, 864-865, 1959.

36. Long, R.S.: "Transfer between non-coplanar elliptical orbits", Astronautica Acta, VI, 167-178, 1960.

37. Ting, L.: "Optimum orbital transfer by several impulses", Astronautica Acta, VI, 256-265, 1960.

38. Plimmer, R.N.: "Fuel requirements for inter-orbital transfer of a rocket", 10th International Astronautical Congress Vienna, 1960.

---

* See also [93], [94], [95].

39. Hoelker, R.F. and R. Silber: "The bi-elliptical transfer between coplanar circular orbits", Advances in the Ballistic Missile and Space Technology, Vol. 3, Pergamon Press, 1961.

40. Rider L.: "C haracteristic velocity requirements for impulsive thrust transfer between non-coplanar circular orbits", ARS J.31, 345-351, 1961.

41. Lawden, D.F.: "Impulsive transfers between elliptical orbits", Optimization Techniques, Academic Press, Ch. 11, pp. 323-351, 1962.

42. Horner, J.M.: "Optimum impulsive orbital transfers between coplanar orbits", ARS J., 32, 1082-1089, 1962.

43. Eckel, K.: "Optimum transfer between non-coplanar elliptical orbits", Astronautica Acta, VIII, 177-192, 1962.

44. Eckel, K.: "Optimum transfer in a central force field with n impulses", Astronautica Acta, IX, 302-324, 1963.

45. Horner, J.M.: "Minimum impulse orbital transfers", AIAA J., 1, 1707, 1963.

46. Altman, S.P. and J.S. Pistiner: "Minimum velocity increment solution for two-impulse coplanar orbital transfer", AIAA J., 1, 435-442, 1963.

47. Lubard S.: "Optimum launching to hyperbolic orbits by two impulses", AIAA J. 1, 2858-2859, 1963.

48. Gobetz, F.W.: "Optimum transfers between hyperbolic asymptotes", AIAA J., 1, 2034, 1963.

49, Barrar, R.B.: "Two-impulse transfer vs one-impulse transfer : Analytic Theory", AIAA J., 1, 65-68, 1963.

50. Gentry L.: "An analysis of two-impulse orbital transfer", AIAA J., 2, 1767-1773, 1964.

51. Lubard S.: "Optimum launching to hyperbolic orbit by two impulses of unequal specific impulse", Astronautica Acta, X, 138-151, 1964.

52. Vinh, N.X.: "Optimum cotangential transfer between elliptical orbits", AIAA Paper No. 65-64. AIAA 2nd Aerospace Sciences Meeting, New York, January 25-27, 1965.

53. Zakoteeva, L.V. and V.V. Polyachenko: "Optimal three-impulse rotation of the plane of a circular orbit with application of thrust impulses on crossing nodal line", Soviet Engineering Journal, Vol. 5, No. 1, Jan.-Feb., 1965.

54. McCue, G.A. and D.F. Bender: "Numerical investigation of minimum impulse orbital transfer", AIAA J., 3, 2328-2334, 1965.

55. Pierucci, M.: "Optimum orbital transfer by two impulses of unequal specific impulse", Astronautica Acta, 11, 269-270, 1965.

56. Hiller, H.: "Optimum transfers between non-coplanar circular orbits", Planet/26 Space Science 13, 147-161, Pergamon Press, 1965.

57. Hiller, H.: "Optimum impulsive transfers between elliptic and non-coplanar circular orbits", Planet. Space Sciences, 13, 1233-1247, 1965.

58. Gunther, P.: "Asymptotically optimum two-impulse transfer from lunar orbit", AIAA J., 4, 346-352, 1966.

59. Bender, D.F. and G.A. McCue: "Conditions for optimal one-impulse transfer", The Journal of Astronautical Sciences, 13, 153-158, 1966.

60. Hiller, H.: "Optimum impulsive transfers between non-coplanar elliptic orbits having collinear major axes", Planet. Space Sciences, 14, 773-789, 1966.

61. Bender D.F. and G.A. McCue: "Optimum transfers between nearly tangent orbits", The Journal of Astronautical Sciences, 13, 72-75, 1966.

62. Moyer, H.G.: "An analytic treatment of cotangential transfer", AIAA J., 5, 1197-1198, 1967.

63. Breakwell, J.V.: "Minimum impulse transfer", AIAA Preprint 63-416, August, 1963. Also AIAA Progress in Astronautics and Aeronautics : Celestial Mechanics and Astrodynamics, Academic Press, Inc., Vol. 14, pp. 583-589, New York, 1964.

64. Marec, J.P.: "Economical Orbital Transfers. Coaxial Coplanar Elliptical Orbits. Unlimited Time.", La Recherche Aerospatiale, No. 105, pp. 11-21, March-April, 1965.

65. Marchal, C.: "Optimal transfers between elliptical orbits (Undetermined Time)", 16th International Congress of Astronautics, Athens, 12-18 September, 1965.

66. Marchal, C.: "Optimal Transfers Between Coplanar Elliptical Orbits (Undetermined Time)", Astronautica Acta, 11, 432-445, 1965.

67. Moyer, H.G.: "Minimum impulse coplanar circle-ellipse transfer", AIAA J., 3, 723-726, 1965.

68. Moyer, H.G.: "Necessary conditions for optimal single impulse transfer", AIAA J., 4, 1405-1410, 1966.

69. Marchal, C.: "Optimal transfers between hyperbolic orbits (Nonzero planetary radius)", 17th International Congress of Astronautics, Madrid, 9-15 October, 1966.

70. Gobetz, F.W., Washington, M. and T.N. Edelbaum: "Minimum-impulse time-free transfer between elliptic orbits", United Aircraft Research Laboratories Report No. 910330-4, April, 1966.

71. Winn, C.B.: "Minimum-fuel transfers between coaxial orbits, both coplanar and non-coplanar", AAS Preprint 66-119, July, 1966.

71 bis. Winn, C.B.: "Minimum fuel transfers between coaxial orbits, both coplanar and non-coplanar", Final Report US Air Force Research Contract, AF (615) - 141.

72. Culp, R.D.: "Contensou-Busemann condition for optimal coplanar orbit transfer", AIAA J., 5, 371-372, 1967.

73. Breakwell, J.V.: "Minimum-impulse transfer between a circular orbit and a nearly non-coplanar elliptic orbit", Colloquium on Advanced Problems and Methods for Space Flight Optimization, Liege, June 19-23, 1967. /27

74. Marchal, C.: "Synthesis of the Analytical Results on Optimal Transfers Between Keplerian Orbits (Undetermined Time)", Colloquium on the Problems and Methods Advanced for the Optimization of Space Flights, Liege, 19-23 June, 1967.

75. Marchal, C.: "Optimal Transfers between Elliptical Orbits (Undetermined Time)", PH.D. Thesis, A01609, Paris School of Science, 1967.

76. Ross, S.: "Optimal ascent into orbit : a new look at an old problem", 15th International Astronautical Congress, Varsovie, September, 1964.

77. Marchal, C.: "Optimization of the Extra-atmospheric Phase of Ascent into Orbit", La Recherche Aerospatiale, No. 116 and 118, 1967.

Fixed Time

Close orbits.

78. Edelbaum, T.N.: "Propulsion requirements for controllable satellites", ARS J., Vol. 31, No. 8, August, 1961.

79. Burt, E.G.C.: "On space manuevers with continuous thrust", RAE T.R. No. 66149, May, 1966.

80. Ross, S. and G. Leitmann: "Low acceleration trajectory optimization in a strong central force field", Proceedings of the IAS Symposium on Vehicle Systems Optimization., Institute of Aerospace Sciences, pp. 127-137, New York, 1961.

81. Edelbaum, T.N.: "Optimum low-thrust rendezvous and station keeping", AIAA J. 2, 1196-1201, 1964.

82. Gobetz, F.W.: "Optimal variable-thrust transfer of a power-limited rocket between neighboring circular orbits", AIAA J, 2, 339-343, 1964.

83. Gobetz, F.W.: "A linear theory of optimum low-thrust rendezvous trajectories", The Journal of the Astronautical Sciences 12, 69-76, 1965.

84. Rider, L.: "Low thrust correction of orbital orientation", ARS J., Vol. 30, No. 7, July 1960.

85. Hinz, H.K.: "Optimal low-thrust near-circular orbital transfer", AIAA J., 1, 1367-1371, 1963.

86. McIntyre, J.E.: "Optimal transfer of a thrust-limited vehicle between coplanar orbits", Ph.D. Thesis, Department of Aerospace and Mechanical Sciences, Princeton University, Princeton, N.J., USA, 1964.

87. Marec, J.P.: "Economical Transfers Between Infinitely Close Orbits", Astrodynamics (Gauthier-Villars, Paris 1966). Communication Presented to the 16th Internationsl Congress of Astronautics, Athens, 1965.

88. McIntyre, J.E. and L. Crocco: "Linearized treatment of the optimal transfer of a thrust-limited vehicle between coplanar circular orbits", Astronautica Acta 12, 224-234, 1966.

89. McIntyre, J.E. and L. Crocco: "Higher order treatment of the optimal Transfer of thrust-limited vehicle between coplanar circular orbits", Astronautica Acta 13, 1967.

90. Marec, J.P.: "Economical Plane Hohmann-type Transfers Between Nearly Circu- /28
lar Coaxial Close Orbits (Fixed Time, Limited Thrust)", Astronautica Acta, 13, 1967.

91. Marec, J.P.: "Optimal Transfers Between Close Elliptical Orbits", Ph.D. Thesis, Paris School of Sciences, No. A01610, 1967.

92. Kuzmac, G.E., Lavrenko, N.J., Isaev, V.K. and V.V. Sonin: "The linearized theory of multi-impulsive transfers. The problem optimal rocket flight", 15th International Astronautical Congress, Varsovie, September 1964.

93.* Edelbaum, T.N.: "A general solution for minimum impulse transfers in the near vicinity of a circular orbit", NAS 12-26 Final Report, 1966.

94.* Marec, J.P.: "Economical Infinitesimal Impulse Transfers Between Nearly Circular Noncoplanar Orbits". Communication to Comptes Rendus, Academy of Sciences,16 May 1966; also: 17th International Congress on Astronautics, Madrid, 1966; also: ONERA publication No. 115, 1966.

94 bis. Marec, J.P.: "Long-lasting Optimal Impulse Rendezvous Between Nearly Circular Close Orbits, Coplanar or Not". Liege Colloquium on "Advanced Problems and Methods for Space Flight Optimization", Liege, June 19-23, 1967.

95.* Edelbaum, T.N.: "Minimum impulse transfers in the near vicinity of a

* More properly pertaining to the case of undetermined time.

95. cont....
circular orbit". The Journal of the Astronautical Sciences, 14, 66-73, 1967.

96. Clohessy, W.R. and R.S. Wiltshire: "Terminal guidance system for satellite rendezvous". J. Aerospace Sciences, 27, 653, 1960.

97. London, H.S.: "Second approximation to the solution of rendezvous equations" AIAA J., 1, 1691-1693, 1963.

98. Kochi, K.C.: "Exact first-order navigation-guidance mechanization and error propagation equations for two-body reference orbits". AIAA J., 2, 365--366, 1963.

99. Niemi, N.J.: "Investigation of terminal guidance system for a satellite rendezvous". AIAA J., 1, 405-411, 1963.

100. Greene, A.H. and W.F. Jaros: "Comparison of error transfer matrices for circular orbits". AIAA J., 1, 2623-2625, 1963.

101. Leach, R.: "Matrix derivation of a short-term linear rendezvous equation". AIAA J., 1, 1420-1421, 1963.

102. De Vries, J.P.: "Elliptic elements in terms of small increment of position and velocity components". AIAA J., 1, 2626-2629, 1963.

103. Billik, B.H.: "Some optimal low-acceleration rendezvous maneuvers". AIAA J. 2, 510-516, 1964.

104. Hempel, P. and J. Tschauner: "Acceleration Programs of Minimal Transfer Energy for the Rendezvous Maneuver". Astronautica Acta, X, 221-237, 1964.

105. Tschauner, J. and P. Hempel: "Optimal Acceleration Programs for the Rendezvous Maneuver". Astronautica Acta, X, 296-307, 1964.

106. Tschauner, J. and P. Hempel: "Rendezvous with a Target Describing an Elliptical Trajectory". Astronautica Acta, XI, 104-109, 1965.

107. Tschauner, J.F.A. and P.R. Hempel: "Minimum-fuel rendezvous techniques". /29
Journal of Spacecraft and Rockets, 2, 802-804, 1965.

108. Tschauner, J.: "New Presentation of Rendezvous with Elliptical Target Trajectory". Astronautica Acta, 11, 312-321, 1965.

109. Anthony, M.L. and F. T. Sasaki: "Rendezvous problem for nearly circular orbits". AIAA J., 3, 1666-1673, 1965.

110. Hahn, D.W. and B.F. Itzen: "Optimum co-altitude rendezvous trajectories with continuous thrust". AIAA J., 3, 2151-2152, 1965.

111. Hempel, P.: "A Reduction of the Rendezvous Maneuver to Trajectory Transfers". Astronautica Acta, 12, 153-158, 1966.

112. Hempel, P.: "Optimal Three-dimensional Ascent Trajectory in a Model Gravitational Field". Zeitschrift fur Flugwissenschaften, 15, 85-90, 1967.

113. Tschauner, J.: "Elliptic orbit rendezvous". AIAA J., 5, 1110-1113, 1967.

114. Euler, E.A. and Y. Shulman: "Second-order solution to the elliptical rendezvous problem". AIAA J., 5, 1033-1035, 1967.

115. Moskowitz, S.E.: "On the accuracy of approximate thrust steering schedules in optimal correctional maneuvers". Astronautica Acta, 9, 20-30, 1963.

116. Platonov, A.K.: "Investigation of the correctional maneuver peculiarities for space flights". 16th International Astronautical Congress, Athens, 1965.

117. Platonov, A.K., Dashkov, A.A. and V.N. Kubasov: "Optimization of spacecraft flight control". IFAC Symposium on Automatic Control in the Peaceful Uses of Space. Stavanger, Norway, June, 1965.

118. Stern, R.G. and J.E. Potter: "Optimization of midcourse velocity corrections". IFAC Symposium on Automatic Control in the Peaceful Uses of Space. Stavanger, Norway, June, 1965.

119. Newell, D.H.: "Spin stabilized interplanetary spacecraft navigation". 17th International Astronautical Congress, Madrid, 9-15 October, 1966.

120. Platonov, A.K. and U.D. Teterin: "Some peculiarities of the two-component non-identical solar correction". 17th International Astronautical Congress, Madrid, 9-15 October, 1966.

121. Lawden, D.F. and R.S. Long: "The theory of correctional manuevers in interplanetary space". Astronautica Acta VI, 48-60, 1960.

122. Lawden D.F.: "Optimal program for correctional manuevers". Astronautica Acta VI, 195-205, 1960.

123. Battin, R.H.: "A statistical optimizing navigation procedure for space flight". ARS J, 32, 1681-1602, 1962.

124. Breakwell, J.V.: "The optimum spacing of corrective thrusts in interplanetary navigation". Mathematics in Science and Engineering, Academic Press Inc., Vol. 5, Chapter 12, New York, 1962.

125. Denham, W.F. and J.L. Speyer: "Optimal measurement and velocity correction programs for midcourse guidance". AIAA J., 2, 896-907, 1964.

126. Breakwell, J.V., Tung, F., and R.R. Smith: "Application of the continuous and discrete strategies of minimum effort theory to interplanetary guidance". AIAA J., 3, 907-912, 1965.

127. Tokarev, V.V.: "Optimal correction program for motion in a gravitational field under restricted power". IFAC Symposium on Automatic Control in the Peaceful Uses of Space. Stavanger, Norway, June, 1965. /30

128. Pfeiffer, C.G.: "A dynamic programming analysis of multiple guidance corrections of a trajectory". AIAA J., 3, 1674-1681, 1965.

129. Abolghassem, Ghaffari: "Multiple midcourse maneuvers in interplanetary guidance". NASA X 507 66 254, June 1966.

Distant Orbits

130. Lancaster, E.R., Blanchard, R.C. and R.A. Devaney: "A note on Lambert's theorem". Journal of Spacecraft and Rockets, 3, 1436- , 1966.

130 bis. Clarke, V.C., Bollman, W.E. and W.J. Scholey: "Design parameters for ballistic interplanetary trajectories, Part I: one way transfer to Mars and Venus". Jet Prop. Lab., Pasadena, California, TR 32-77, March 1, 1962.

131. McCue, G.A.: "Optimum tow-impulse orbital transfer and rendezvous between inclined elliptical orbits". AIAA J., 1, 1865-1872, 1963.

132. Collins, R.L. and S.A. Wallace: "A computational method for two-impulse orbital rendezvous and transfer problems". NASA T.N D-3508, 1966.

133. Eckel, K.G.: "Expense verses time in orbital transfer to rendezvous". 17th International Astronautical Congress, Madrid, 1966.

134. Hayes, W.E. and W.W. Hinton: "Planning a first orbit rendezvous mission". 17th International Astronautical Congress, Madrid, 1966.

135. Koenke, E.J.: "Minimum two-impulse transfer between coplanar circular orbits". AIAA Paper No. 66-11. AIAA 3rd Aerospace Sciences Meeting, New York, January 24-26, 1966.

136. Lee, G.: "An analysis of two-impulse orbital transfer". AIAA J., 2, 1767, 1964.

137. Faulders, C.R.: "Optimum thrust programming of electrically powered rockets in a gravitational field". ARS J., Vol. 30, No. 10, October, 1960.

138. Edelbaum, T.N.: "Optimum low-thrust transfer between circular and elliptic orbits". Proceedings of the Fourth US National Congress on Applied Mechanics, American Society of Mechanical Engineers, pp. 137-141, New York, 1962.

139. Edelbaum, T.N.: "Theory of maxima and minima". Optimization Techniques, edited by G. Leitmann, Ch. 1, pp. 19-30, Academic Press, New York, 1962.

140. Edelbaum, T.N.: "Optimum power-limited orbit transfer in strong gravity fields". AIAA J., 3, 921-925, 1965.

141. Smith, F.T.: "The optimization of interplanetary orbit transfers by dynamic programming". Memorandum RM-4622-PR, The Rand Corporation, July 1965.

142. Edelbaum, T.N.: "An asymptotic solution for optimum power limited orbit transfer". AIAA J., 4, 1491-1494, 1966.

143. Melbourne, W.G.: "Interplanetary trajectories and payload capabilities of advanced propulsion vehicles". Jet propulsion Lab. TR 32-68, 1961.

144. Irving, J.H.: "Low thrust flight : variable exhaust velocity in gravitational fields". Space Technology, Ch. 10, John Wiley and Sons.

145. Saltzer, C. and C.W. Fetheroff: "A direct variational method for calcula- /31
tion of optimum thrust programs for power-limited interplanetary flight". Astronautica Acta, 7, 8-20, 1961.

146. Melbourne, W.G.: "Three-dimensional optimum thrust trajectories for power limited propulsion systems". ARS J., 1723-1728, December, 1961.

147. Melbourne, W.G. and C. G. Sauer: "Optimum thrust programs for power-limited propulsion systems". Astronautica Acta, 8, 205-227, 1962.

148. Melbourne, W.G. and C. G. Sauer: "Optimum interplanetary rendezvous with power-limited vehicles". AIAA J. 1, 54-61, 1963.

149. Stancil, R.T.: "A new approach to steepest-ascent trajectory optimization". AIAA J. 2, 1365-1370, 1964.

150. Ivanov Ju.N. and Ju.V. Salaev: "Application de la methode du gradient au calcul des trajectoires interorbitales pour des propulseurs a puissance limitee". Kosmiuceskie issledovanija 2, 433, 1964.

151. Pokrovskaya: "Heliocentric Flight with Constant Thrust Using the Method of Gradients". Recherche Cosmique, V, 859, 1964.

152. Girard, H.: "Application of the Gradient Method to the Numerical Study of Limited Thrust Transfers Between Circular Coplanar Close Orbits. Comparison with the Linearized Solution". N. T. Onera, No. 112, 1967.

153. Melbourne, W.G. and C.G. Sauer: "Constant-attitude thrust program optimization". AIAA J., 3, 1428-1431, 1965.

154. Zee, C.H.: "Effect of finite thrusting time in oribtal maneuvers". AIAA J. 1, 60-64, 1963.

155. Wang, K.: "Estimate of effect of large thrust on Hohmann type transfers". ARS J., 32, 642-643, 1962.

156. Zimmerman, A.V., McKay, J.S. and L.G. Rossa: "Optimum low-acceleration trajectories for interplanetary transfers". NASA TN D.1456, January 1963.

157. Graham, R.G.: "A steepest-ascent solution of multiple-arc optimization problems". AIAA J., 3, 154-155, 1965.

158. Eckenwiler, M.W.: "Closed-form Lagrangian multipliers for coast periods of optimum trajectories". AIAA J., 3, 1149-1151, 1965.

159. Handelsman, M.: "Some necessary conditions for optimal fixed-time powered transfers with multiple coast and thrusts between circular orbits". 17th International Astronautical Congress Madrid, October, 1966.

160. Handelsman, M.: "Optimal free-space fixed thrust trajectories using impulsive trajectories as starting iteratives". AIAA J., 4, 1077-1082, 1966.

161. Geerts, J.: "Numerical integration of plane orbital transfers with multiple powered arcs". Colloquium on Advanced Problems and Methods for Space Flight Optimization, Liege, June 19-23, 1967.

162. Pines, S. and T.C. Fang: "A uniform closed solution of the variational equations for optimal trajectories during coast". Colloquium on Advanced problems and Methods for Space Flight Optimization, Liege, June 19-23, 1967.

163. Zee, C.H.: "Powered flight trajectories of rockets under oriented constant thrust". AIAA J., 1, 602-606, 1963.

164. Tsien, H.S.: "Take-off from satellite orbit". Jet Propulsion, 23, 233, /32
1953.

165. Copeland, J.: "Interplanetary trajectories under low thrust radial acceleration". ARS J., April 1959.

166. Lass, H. and J. Lorell: "Low acceleration take-off from a satellite orbit". ARS J., Vol. 31, No. 1, January 1961.

167. Au, G.: "Corrections for *Interplanetary trajectories under low thrust radial acceleration*". ARS J., 30, 708, 1960.

168. Karrenberg, H.K.: "Note on *Interplanetary trajectories under low thrust radial acceleration*". ARS J., 30, 130-131, 1960.

169. Cohen, M.J.: "Low-thrust spiral trajectory of a satellite of variable mass" AIAA J., 3, 1946-1949, 1965.

170. Moekel: "Trajectories with constant tangential thrust in central gravitational fields". NASA TR R.53, 1959.

171. Benney, D.J.: "Escape from a circular orbit using tangential thrust". Jet Propulsion 28, pp. 167-169, March 1958.

172. Zee, C.H.: "Low constant tangential thrust spiral trajectories". AIAA J., 1, 1581-1583, 1963.

173. Zee, C.H.: "Powered flight trajectories of rockets under constant tangential thrust". The Journal of Astronautical Sciences, 1, 1-6, 1965.

174. Shi, Y.Y. and M.C. Eckstein: "An approximate solution for ascending low-thrust trajectories without singularity". AIAA J., 5, 170-172, 1967.

175. Auelmann, R.R.: "Trajectories with constant normal force starting from a circular orbit". AIAA J., 2, 561-563, 1964.

176. Ting, L. and S. Brofman: "On take-off from circular orbit by small thrust". Z. Angew. Math. Mech., 44, 417-428, 1964.

177. Nayfeh, A.H.: "Take-off from a circular orbit by a small thrust". AIAA Paper 65-688, September, 1965; also AIAA Progress in Astronautics and Aeronautics : Methods in Astrodynamics and Celestial Mechanics, Academic Press Inc., Vol. 17, 139-157, New York, 1966.

178. Shi, Y.Y. and M.C. Eckstein: "Ascent or descent from satellite orbit by low thrust". AIAA J., 4, 2203-2209, 1966.

179. Lawden, D.F.: "Optimal programming of rocket thrust direction". Astronautica Acta, I, 40-56, 1955. Presented at the 5th IAF Congress at Innsbruck, August 6, 1954.

180. Faulders, C.R.: "Low thrust steering program for minimum time transfer between planetary orbits". Memoire Mc65 - 58 - 88A, S. Aut. Eng. Paper, October, 1958.

181. Lawden, D.F.: "Optimal escape from a circular orbit". Astronautica Acta, IV, 218-233, 1958.

182. Long, R.S.: "Escape from a circular orbit with finite velocity at infinity" Astronautica Acta, V, 160-162, 1959.

183. Faulders, C.R.: "Minimum time steering programs for orbital transfer with low thrust rockets". Astronautica Acta, Vol. 7, Fasc. 1, 1961.

184. Lindorfer, W. and H.G. Moyer: "Application of a low thrust trajectory /33
optimization scheme to planar Earth-Mars transfer". ARS J., pp. 260-262, February, 1962.

185. Bleik, W.E.: "Orbital transfer in minimum time". AIAA J., 1, 1229-1231, 1963.

186. Lebedev, V.N. : "Some problems of the optimal transfer theory". 14th International Astronautical Congress, Paris, September 1963.

187. Anthony: "Analytical determination of optimal escape with constant acceleration". AIAA Paper, No. 65-699.

188. Edelbaum, T.N.: "The use of high- and low-thrust propulsion in combination for space missions". Journal of Astronautical Sciences, 9, 58-59, 1962.

Singular Arcs

189. Lawden, D.F. : "Optimal intermediate-thrust arcs in a gravitational field". Astronautica Acta, 8, 106-123, 1962.

190. Kelley, H.J.: "Singular extremals in Lawden's problem of optimal rocket flight". AIAA J., 1, 1597, 1963.

191. Kelley, H.J., Kopp, R.E. and H.G. Moyer: "Singular extremals". Optimization Theory and Applications, Academic Press, Vol. 1A, Variational Approach, Chapter 3.

192. Keller, J.L.: "On minimum propellant paths for thrust limited rockets". Astronautica Acta X, 262-269, 1964.

193. Kopp, R.E. and H.G. Moyer: "Necessary conditions for singular extremals". AIAA Paper No. 65-63; AIAA 2nd Aerospace Sciences Meeting, New York, January 25-27, 1965.

194. Robbins, H.M.: "Optimality of intermediate-thrust arcs of rocket trajectories". AIAA J., 3, 1094-1098, 1965.

195. Robbins, H.M.: "Optimal rocket trajectories with subarcs of intermediate thrust". 17th International Astronautical Congress, Madrid, October 9-15, 1966.

196. Marchal, C.: "Tri-dimensional Generalization and Study of Optimality of Lawden's Intermediate Thrust Arcs. Astronautica Acta - a paraitre.

Transfers Between Planets

197. Lawden, D.F.: "Optimal transfer between circular orbits about two planets". Astronautica Acta, I, 89-99, 1955.

198. Fox, R.H.: "High-energy, low-thrust Jupiter missions". ARS Paper No. 1477--60; ARS 15th Annual Meeting, Shoreham Hotel, Washington, D.C., December 5-8, 1960.

199. Green, B.S. and N. Lewin: "A gradient method for obtaining circumlunar trajectories". AIAA Paper No. 63-401; AIAA Astrodynamics Conference, Yale University, New Haven, Connecticut, USA, August 19-21, 1963.

200. Minovitch, M.A.: "Determination and characteristics of ballistic interplanetary trajectories under the influence of multiple planetary attractions". T.R. No. 32-464, J.P.L. Pasadena, California, October 31, 1963.

201. Luidens, R.W.: "Mars nonstop round-trip trajectories". AIAA J., 2, 368--369, 1964. /34

202. Minovitch, M.A.: "Utilizing large planatary perturbations for the design of deep space, solar probe, and out-of-ecliptic trajectories". TR No. 32-849, J.P.L., Pasadena, California, December 15, 1965.

203. Cutting, E. and F.M. Sturms, Jr.: "Trajectory analysis of a 1970 mission to Mercury via a close encounter with Venus". AIAA Paper No. 65-90; AIAA 2nd Aerospace Sciences Meeting, New York, N.Y., January 25-27, 1965.

204. Hollister, W.M. and J.E. Prussing: "Optimum transfer to Mars via Venus". Astronautica Acta, 12, 169-179, 1966.

205. Flandro, G.A.: "Fast reconnaissance missions to the outer solar system utilizing energy derived from the gravitational field of Jupiter". Astronautica Acta, 12, 329-337, 1966.

206. Narin, F.: "Choice of flight mode for outer planet missions". 17th International Astronautical Congress, Madrid, 1966.

207. Zola, C.L. and L.H. Fishbach: "Minimum propellant consumption round trip trajectories to Mars, for constant-thrust, constant-specific-impulse vehicles with optimum coasting periods". NASA TN D. 3233, 1966.

208. Rosenboum, R., Willwerth, R.E. and W. Chuch: "Powered flight trajectories for lunar and interplanetary transfers". Astronautica Acta, 12, 159-168, 1966.

209. Wilson, Jr., S.W. and J.J. Miller: "Matching planetocentric and heliocentric low-acceleration trajectories". AIAA J., 4, 932-934, 1966.

210. Breakwell, J.V. and H.E. Rauch: "Asymptotic matching in power-limited interplanetary transfers". AAS Preprint 66-114. Space Flight Mechanics Specialist Conference, Denver, Colorado, July 6-8, 1966.

211. Breakwell, J.V. and H.E. Rauch: "Optimum guidance for a low thrust interplanetary vehicle". AIAA J., 4, 693-704, 1966.

Miscellaneous

212. Horner, J.M.: "Optimum orbital transfers". Paper submitted to 1960 ARS Undergraduate Competition, 1960.

213. Van Gelder, A., Betrami, E. and H. Munick: "On minimum time, minimum fuel rendezvous". Meeting of SIAM, November, 1960.

214. Leitmann, G.: "Minimum transfer time of a power-limited rocket". J. Appl. Mech., 28, 171-178, 1961.

215. Moulin, L.: "Approximate analysis of optimum low thrust transfer between coplanar hyperbolic orbits". Von Karman Institute for Fluid Dynamics TN 15, May 1964.

216. Johnson, D.P. and L.W. Stumpf: "Perturbation solutions for low-thrust rocket trajectories". AIAA J., 3, 1934-1936, 1965.

217. King, J.C.: "A natural first approximation to low-thrust trajectories between satellite orbits of the Earth and Moon". AIAA J., 3, 1432-1439, 1965.

218. Goldstein, A.A., Greene, A.H., Johnson, A.T. and T.I. Seidman: "Fuel optimization in orbital rendezvous". AIAA Progress in Astronautics and Aeronautics : Guidance and Control, 11, Academic Press, Vol. 13, pp. 823-844, New York, 1964; also AIAA Preprint 63 354, August, 1963.

219. Munick, H., McGill, R. and G.E. Taylor: "Analytic solutions to several /35
optimum orbit transfer problems". Journal of the Astronautical Sciences, Vol. 7, No. 4, 1960.

220. Paiewonsky, B. and P.J. Woodrow: "The synthesis of optimal controls for a class of rocket steering problems". Paper No. 63-224, presented at the AIAA Summer Meeting, Los Angeles, California, USA, June 1963.

221. Niehoff, J.C.: "Gravity-assisted trajectories ot solar-system targets". Journal of Spacecraft and Rockets, 3, 1351, 1966.

222. O'Brien, R.M. and R.F. Sievers: "Rendezvous guidance with limited engine throttling". Journal of Spacecraft and Rockets, 3, 1450, 1966.

223. Pitkin, E.T.: "A regularized approach to universal orbit variables". AIAA J., 1508, August, 1965.

224. Billik, B.H. and H.L. Roth: "Studies relative to rendezvous between circular orbits". Astronautica Acta, 13, 1967.

225. Stern, R.G.: "Singularities in the analytic solution of the linearized variational equations of elliptical motion". 1st Ann. Meeting and Tech. Display of the Am. Inst. of Aeron. and Astron., Wash., D.C., July 1, 1964.

226. Fiul, A. and H. Braham: "Optimization of vehicles and trajectories for the 24 hour equatorial satellite mission". ARS Preprint 1120-60, 1960.

Translated for the National Aeronautics and Space Administration under Contract No. NASw-1695 by Techtran Corporation, P.O. Box 729, Glen Burnie, Md. 21061